Parameter Estimation

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Central Limit Theorem

The Central Limit Theorem

- Consider I.I.D. random variables X_1, X_2, ...
  - X_i have distribution with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$
  - Let: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
  - Central Limit Theorem: $\text{as } n \to \infty$

Version 1

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

Version 2

$$\sum_{i}^{n} X_i \sim N(n\mu, n\sigma^2)$$

Get Good at Manipulating Normals

$$X \sim N(\mu, \sigma^2)$$

$$aX + b \sim N(a\mu, a^2\sigma^2)$$

$$X \sim N(\mu_1, \sigma_1) \qquad Y \sim N(\mu_2, \sigma_2)$$

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1 + \sigma_2)$$

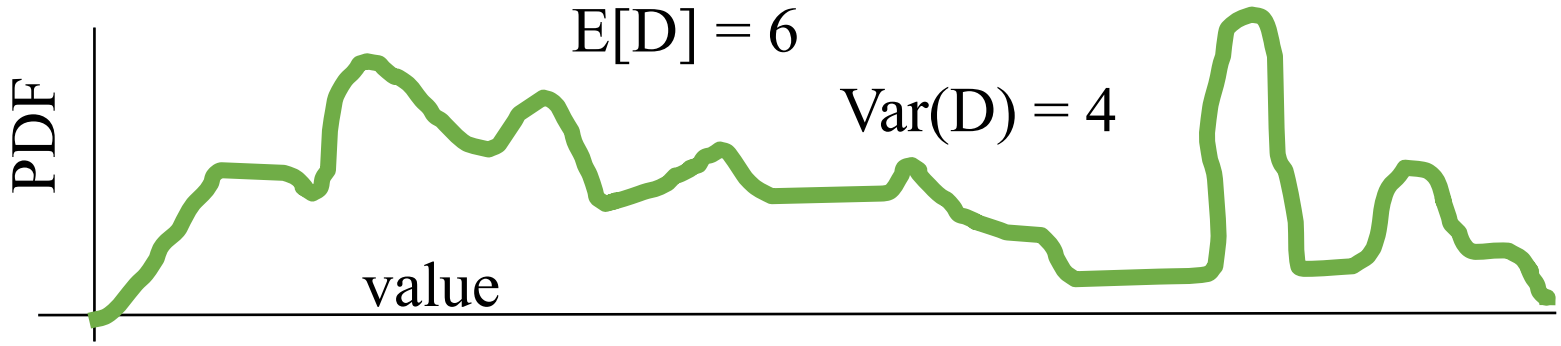
(*If X and Y are independent*)

How To Find Central Limit Theorem?

Are you averaging many (>10) I.I.D. random variables?

Are you adding many (>10) I.I.D. random variables?

You are tracking an object on a 1D line and know its location X. Your radar goes down and you don't get to observe it for 20 time steps. Each time step you assume that its change in position is IID with this this pdf:



What is the distribution of your belief about the location of the object after 20 time steps?

End Review

Warmup: Bounds

Inequality, Probability and Joviality

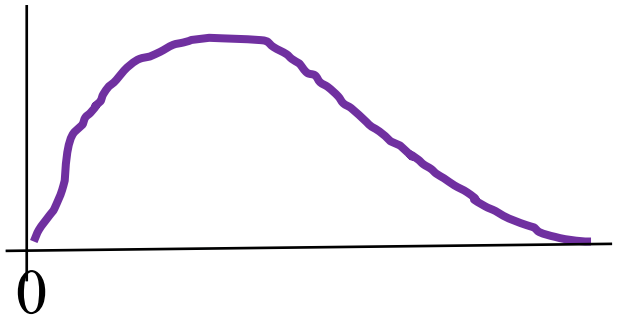
- If we know some statistics of a distribution,
  - E.g., mean, Variance, Non-negativity, Etc.
  - Inequalities and bounds allow us to make analytic claims about the probability distribution
  - May be imprecise compared to knowing true distribution!
  - But a useful tool for proving theorems.

Markov's Inequality

- Say X is a **non-negative** random variable

$$P(X \geq a) \leq \frac{E[X]}{a}, \quad \text{for all } a > 0$$

- Proof:
  - I = 1 if X ≥ a, 0 otherwise
  - Since $X \geq 0, \quad I \leq \frac{X}{a}$
  - Taking expectations:



$$E[I] = P(X \geq a) \leq E\left[\frac{X}{a}\right] = \frac{E[X]}{a}$$

Markov and the Midterm

- Statistics from a previous quarter's CS109 midterm
  - X = midterm score
  - Using sample mean $\underline{X}$ = 84.0 ≈ E[X]
  - What is P(X ≥ 100)?

$$P(X \geq 100) \leq \frac{E[X]}{100} = \frac{84}{100} = 0.84$$

  - Markov bound: ≤ 84% of class scored 100 or greater
  - In fact, 20.1% of class scored 100 or greater
    - Markov inequality can be a very loose bound
    - But, it made <u>no</u> assumption at all about form of distribution!

Andrey Markov

- Andrey Andreyevich Markov (1856-1922) was a Russian mathematician

  - Markov's Inequality is named after him
  - He also invented Markov Chains…
    - …which are the basis for Google's PageRank algorithm
  - John Snow with a mustache?

Chebyshev's Inequality

- X is a random variable with E[X] = μ, Var(X) = σ^2

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}, \quad \text{for all } k > 0$$

- Proof:
  - Since $(X-\mu)^2$ is non-negative random variable, apply Markov's Inequality with $a = k^2$

$$P((X-\mu)^2 \geq k^2) \leq \frac{E[(X-\mu)^2]}{k^2} = \frac{\sigma^2}{k^2}$$

  - Note that: $(X-\mu)^2 \geq k^2 \Leftrightarrow |X-\mu| \geq k$, yielding:

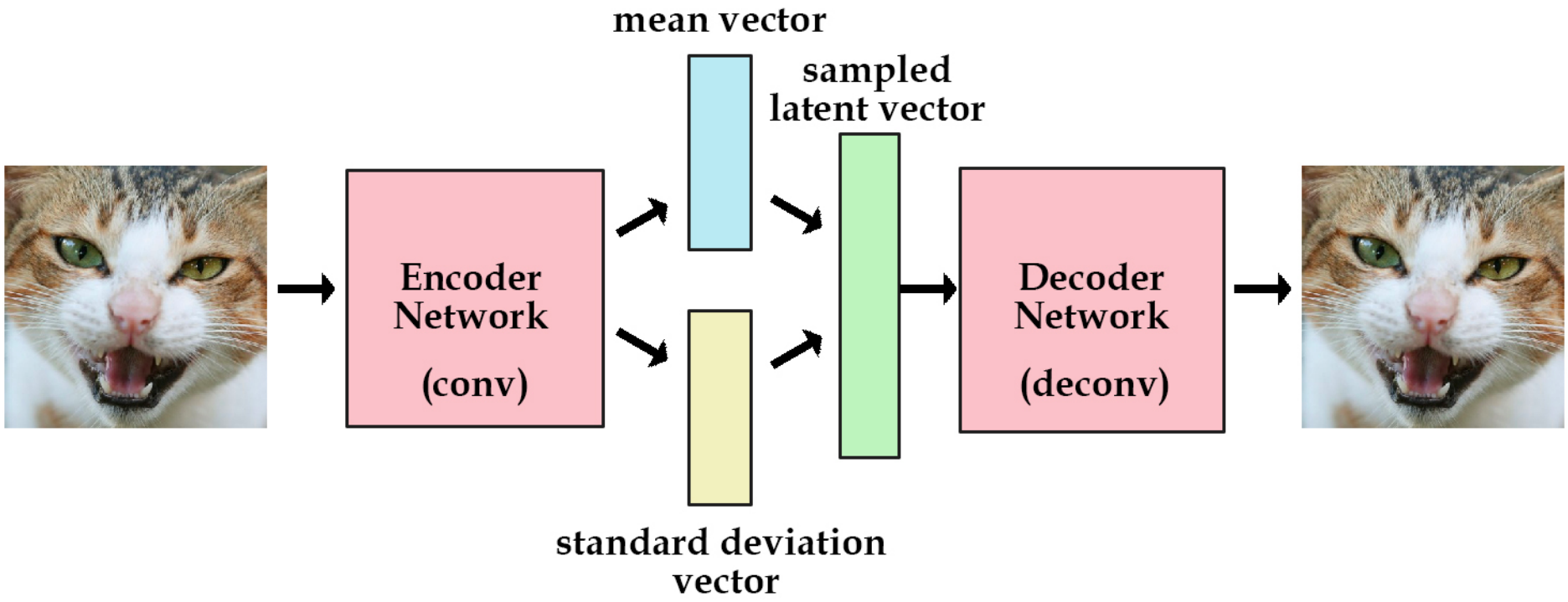
$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Pafnuty Chebyshev

- Pafnuty Lvovich Chebyshev (1821-1894) was also a Russian mathematician

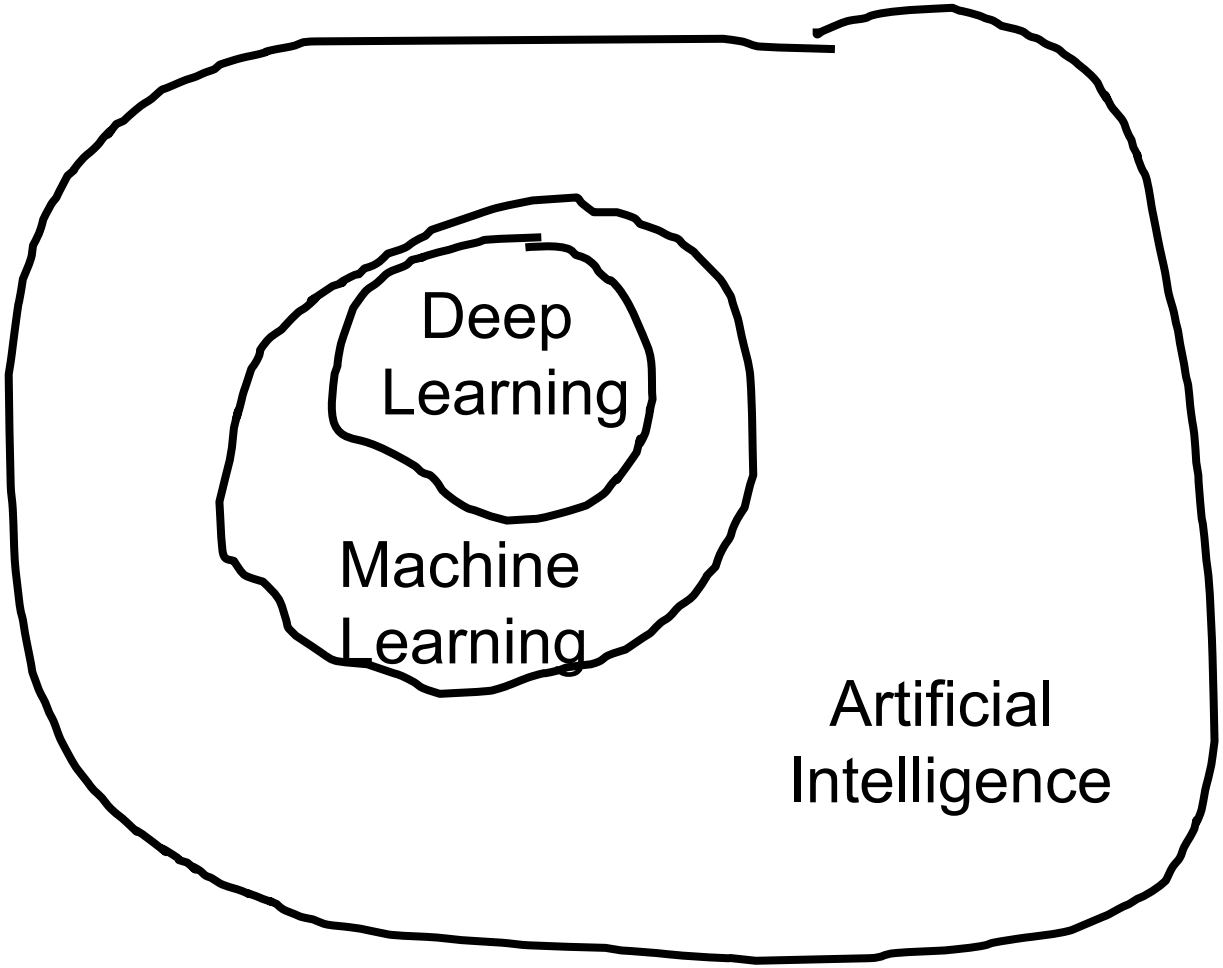
  - Chebyshev's Inequality is named after him
    - But actually formulated by his colleague Irénée-Jules Bienaymé
  - He was Markov's doctoral advisor
    - And sometimes credited with first deriving Markov's Inequality
  - There is a crater on the moon named in his honor

Inequalities for Variational Autoencoders



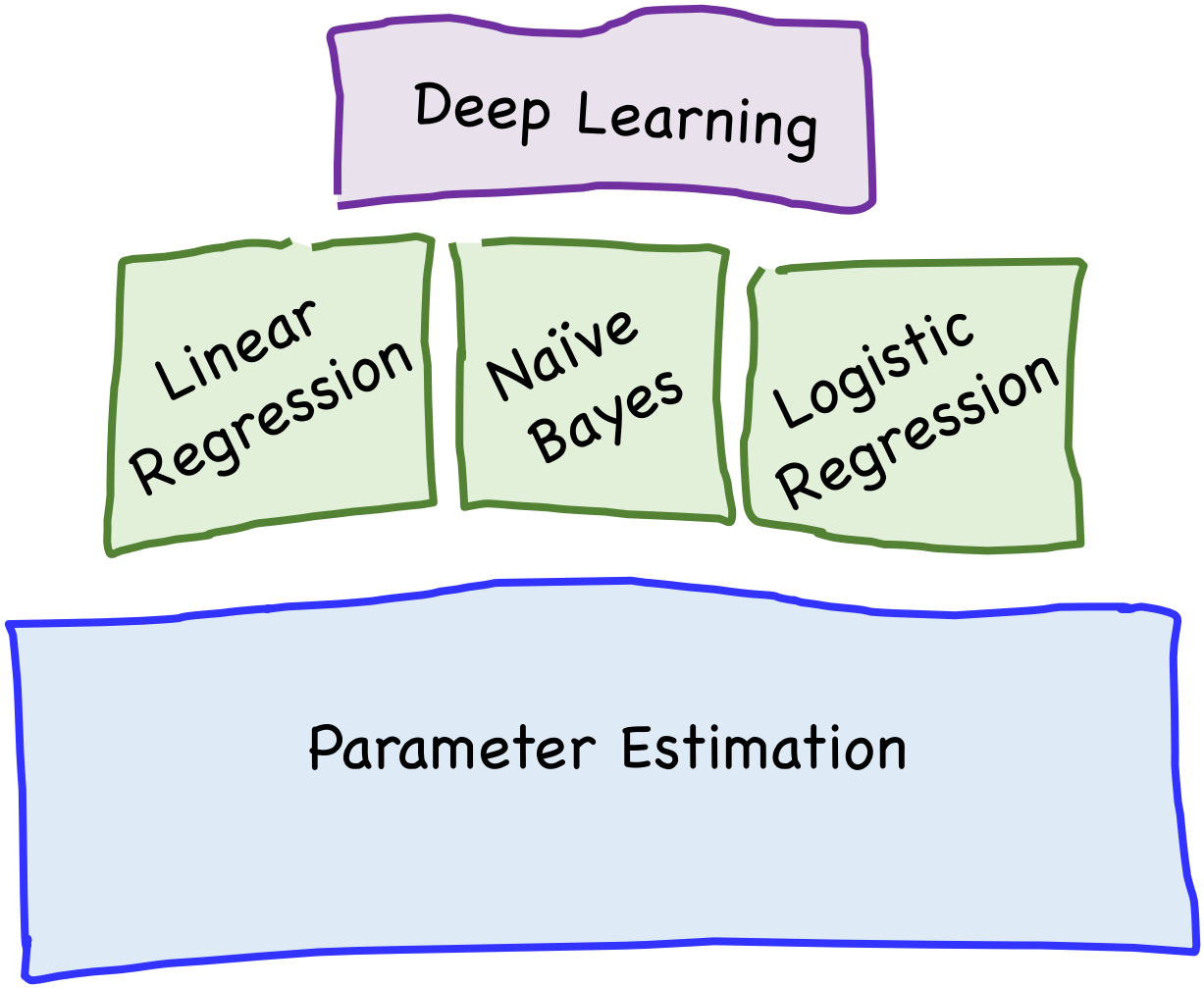
Machine Learning

AI and Machine Learning

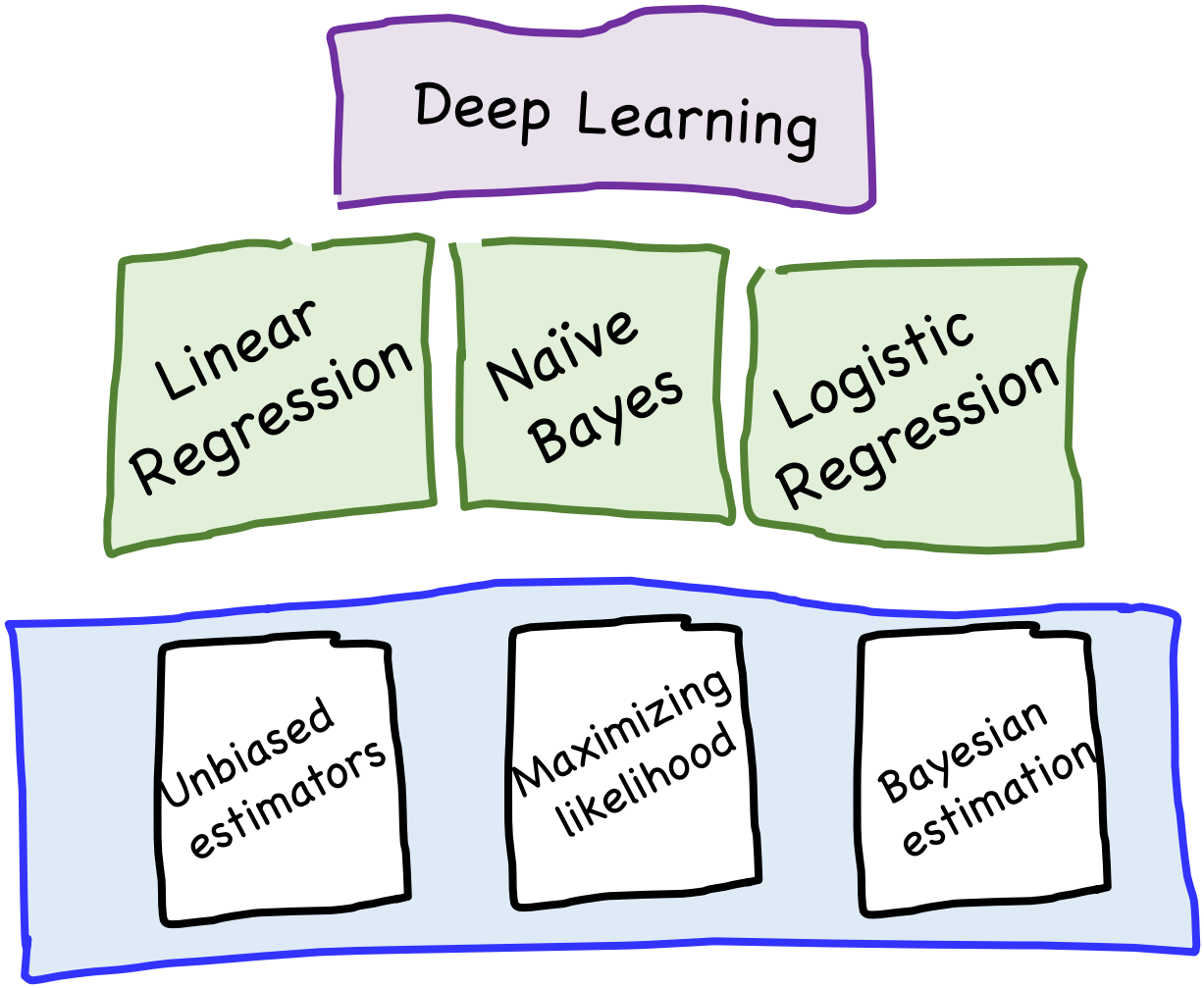


ML: Rooted in probability theory

Our Path

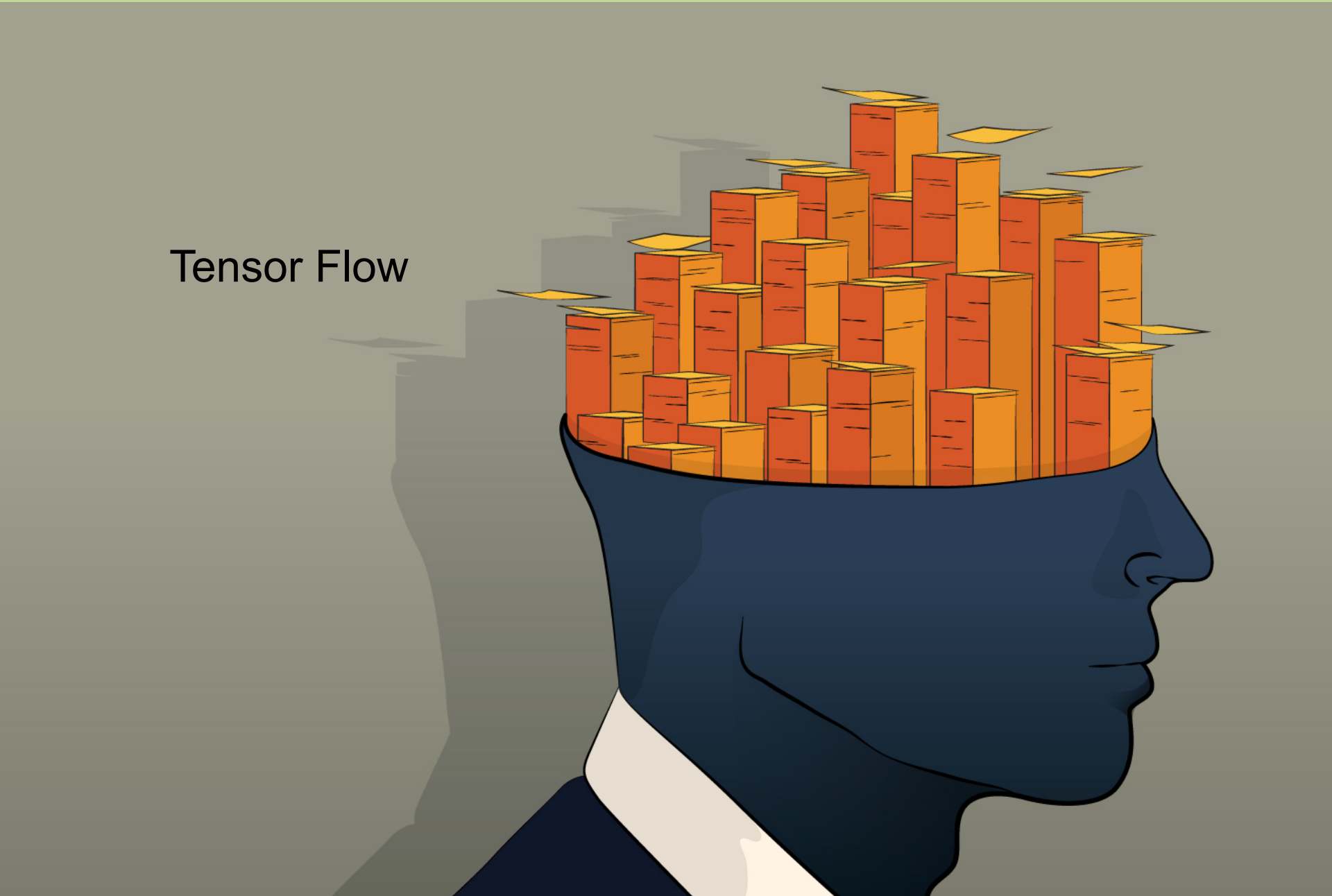

Deep Learning
Linear Regression
Naïve Bayes
Logistic Regression
Parameter Estimation

Our Path


Deep Learning
Linear Regression
Naïve Bayes
Logistic Regression
Unbiased estimators
Maximizing likelihood
Bayesian estimation

Jump Straight to Deep Learning?

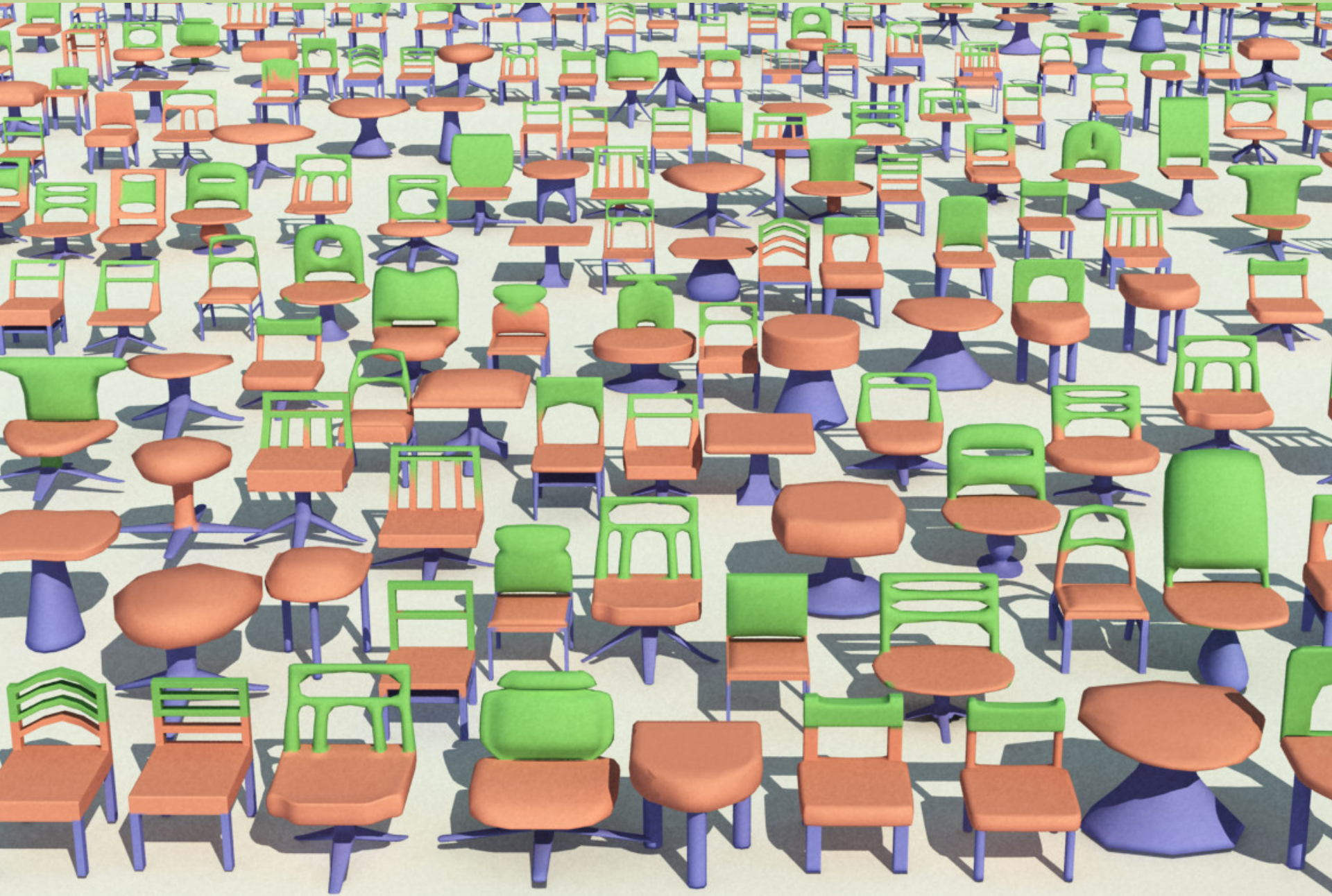
Tensor Flow



Understand the theory to help you debug

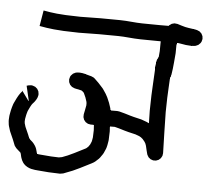
But another reason…

Machine Learning Uses a Lot of Data

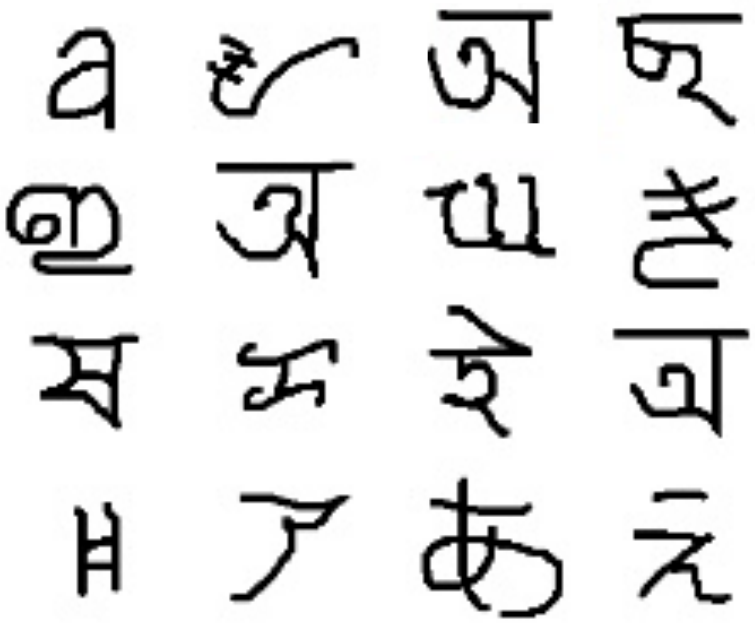


One Shot Learning

Single training example:



Test set:



One Shot Learning

Single training example:

Computers struggle...

Understand the theory to push on the grand challenges


Walt Disney
PICTURES

Once upon a time…

…there was parameter estimation

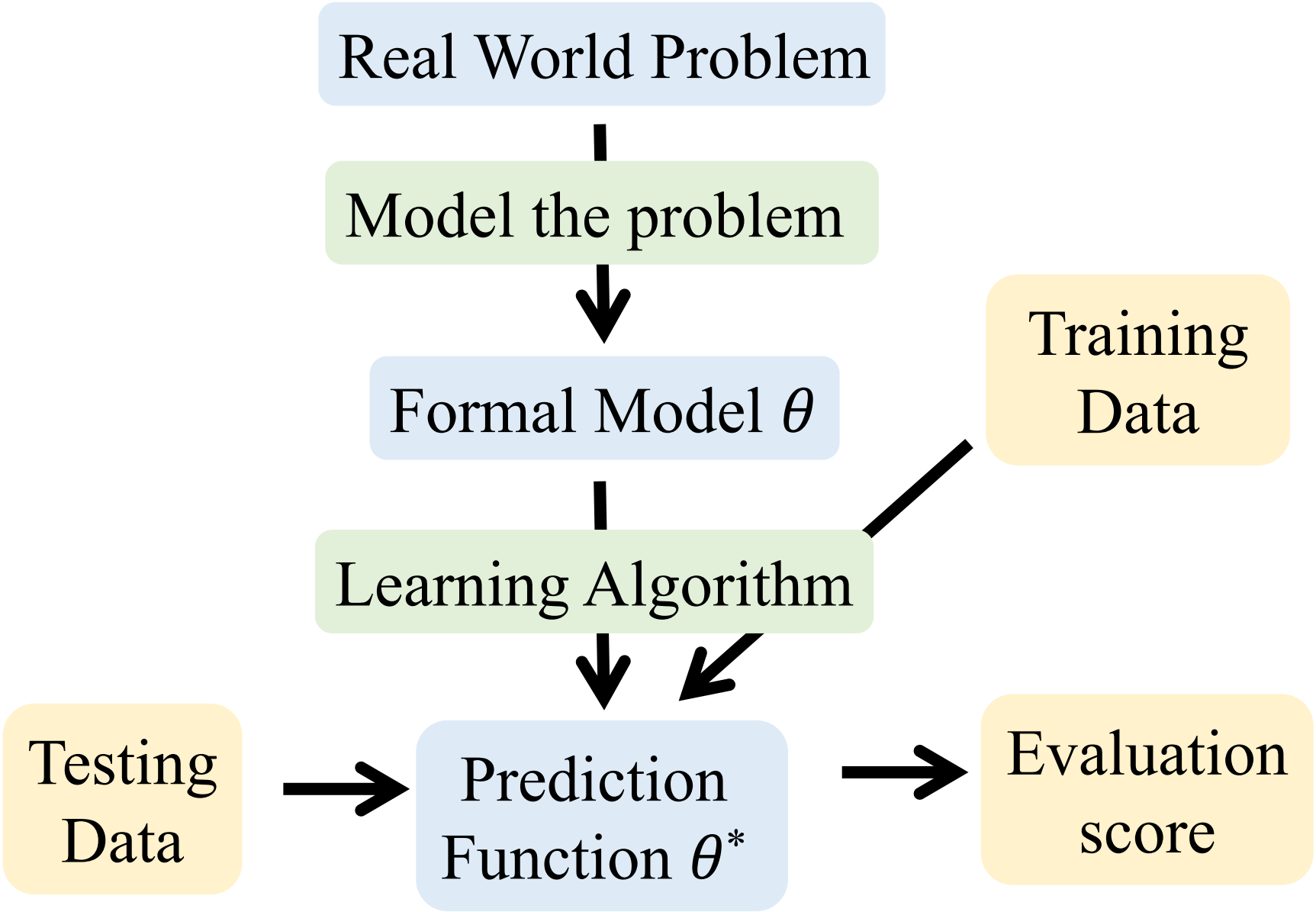
What are Parameters?

- Consider some probability distributions:
  - Ber(p) $\theta = p$
  - Poi(λ) $\theta = \lambda$
  - Uni(α, β) $\theta = (\alpha, \beta)$
  - Normal(μ, σ^2) $\theta = (\mu, \sigma^2)$
  - Y = mX + b $\theta = (m, b)$
  - etc…
- Call these “parametric models”
- Given model, parameters yield actual distribution
  - Usually refer to parameters of distribution as θ
  - Note that θ that can be a vector of parameters

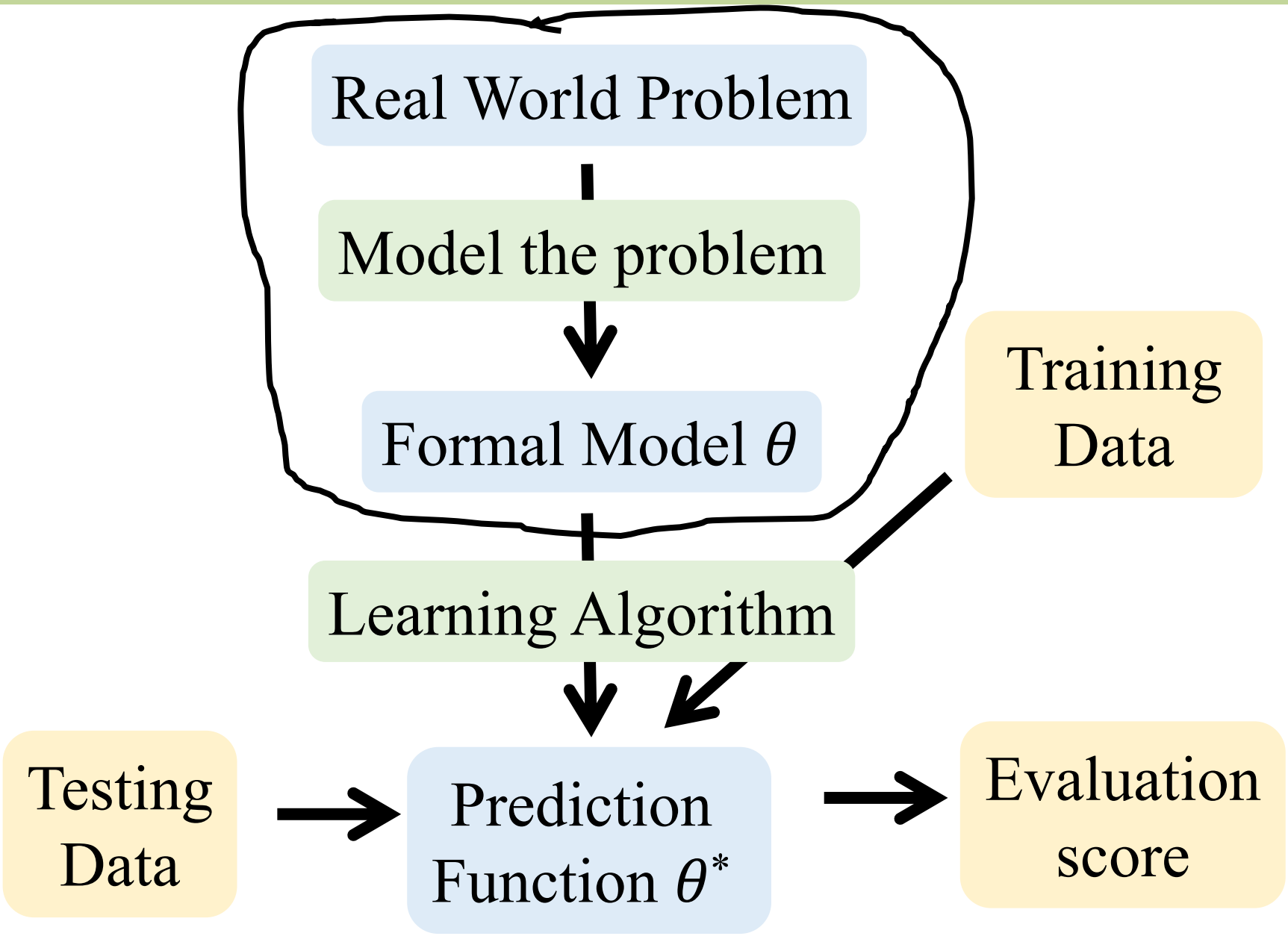
Why Do We Care?

- In real world, don't know "true" parameters
  - But, we do get to observe data
    - E.g., number of times coin comes up heads, lifetimes of disk drives produced, number of visitors to web site per day, etc.
  - Need to estimate model parameters from data
  - "Estimator" is random variable estimating parameter
- Estimate of parameters allows:
  - Better understanding of process producing data
  - Future predictions based on model
  - Simulation of processes

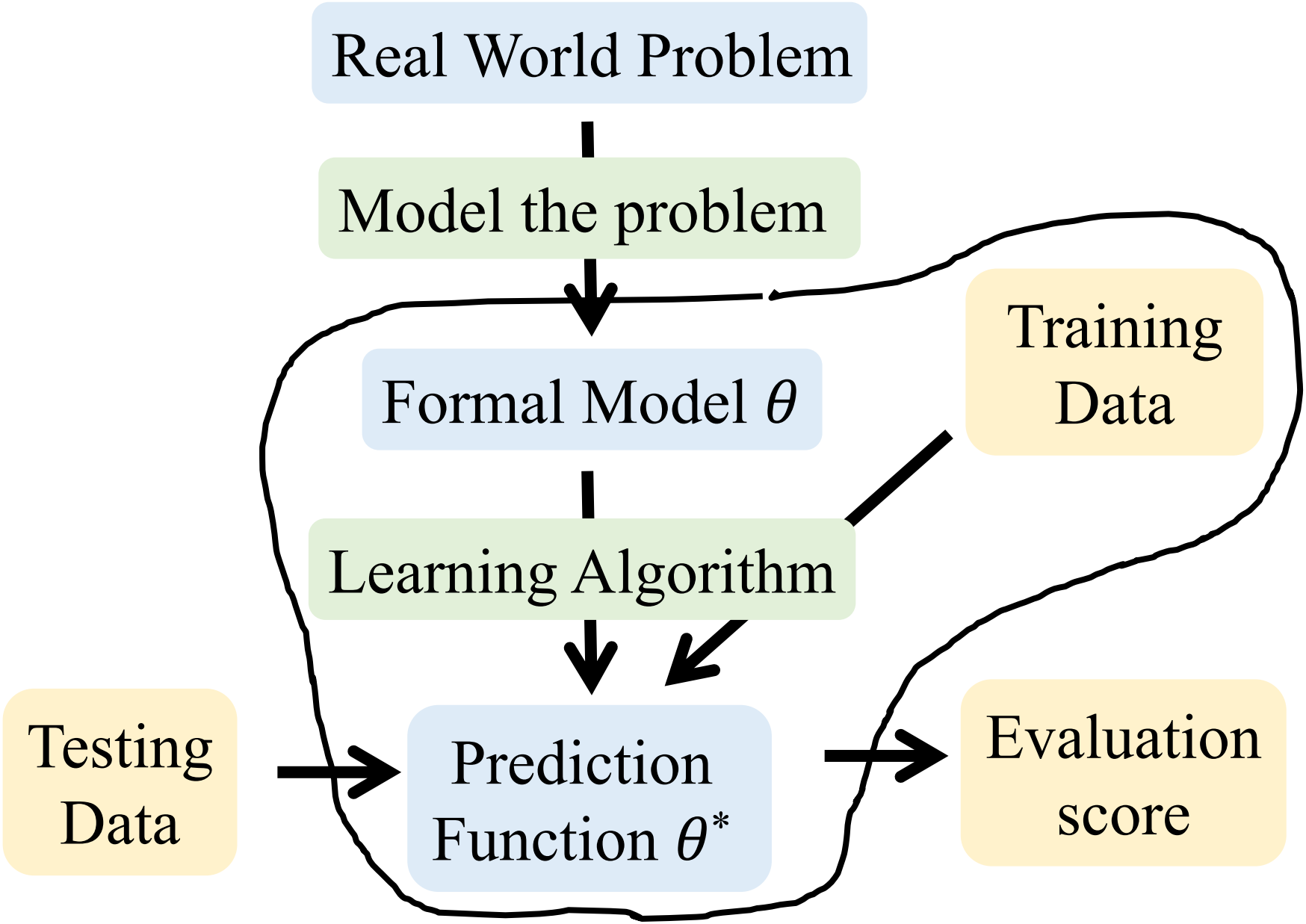
Supervised Learning



Modelling


Real World Problem
Model the problem
Formal Model θ
Training Data
Learning Algorithm
Testing Data
Prediction Function θ^*
Evaluation score

Training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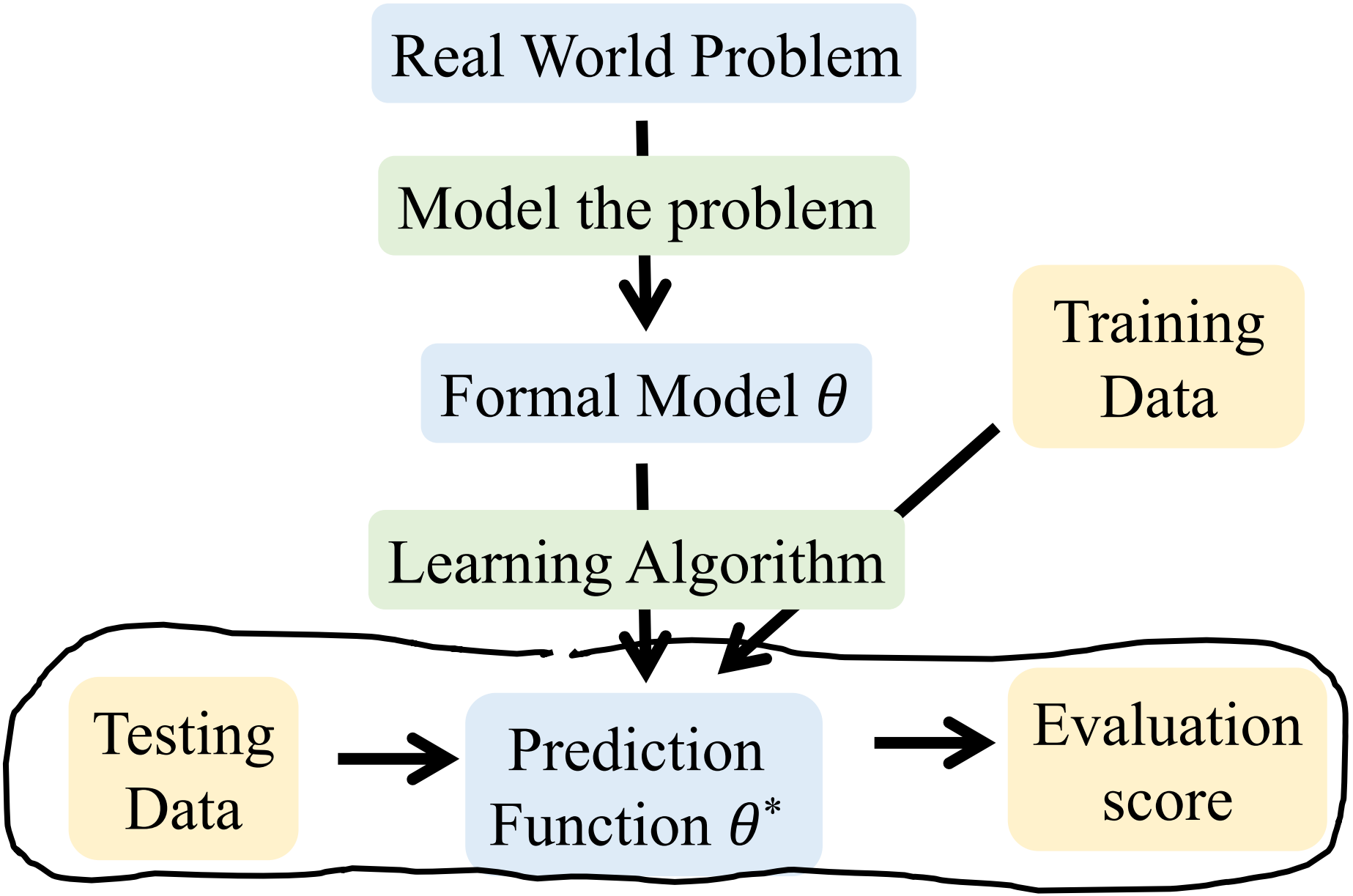
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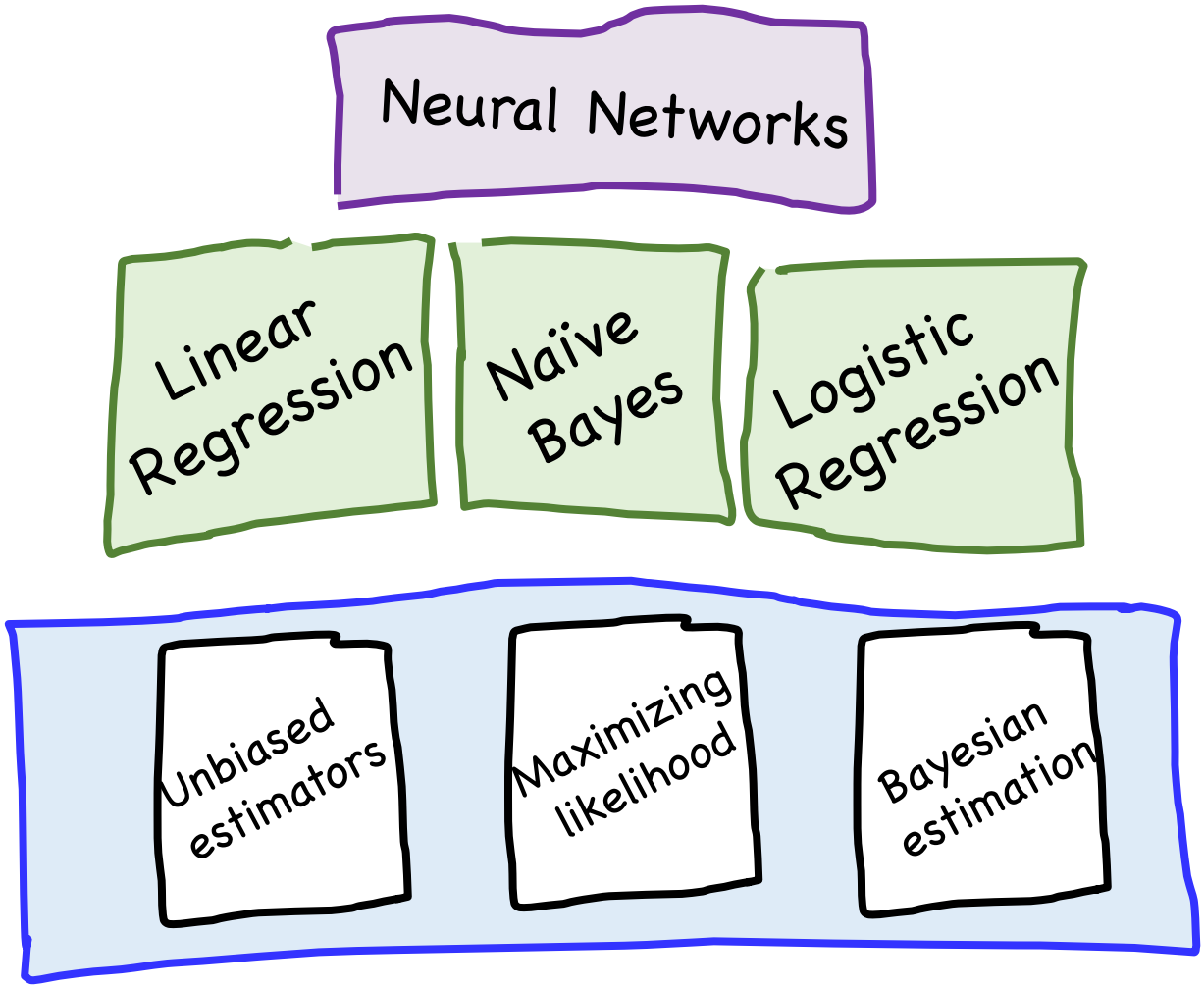
Basis for learning from data

Our Path


Neural Networks
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Unbiased estimators
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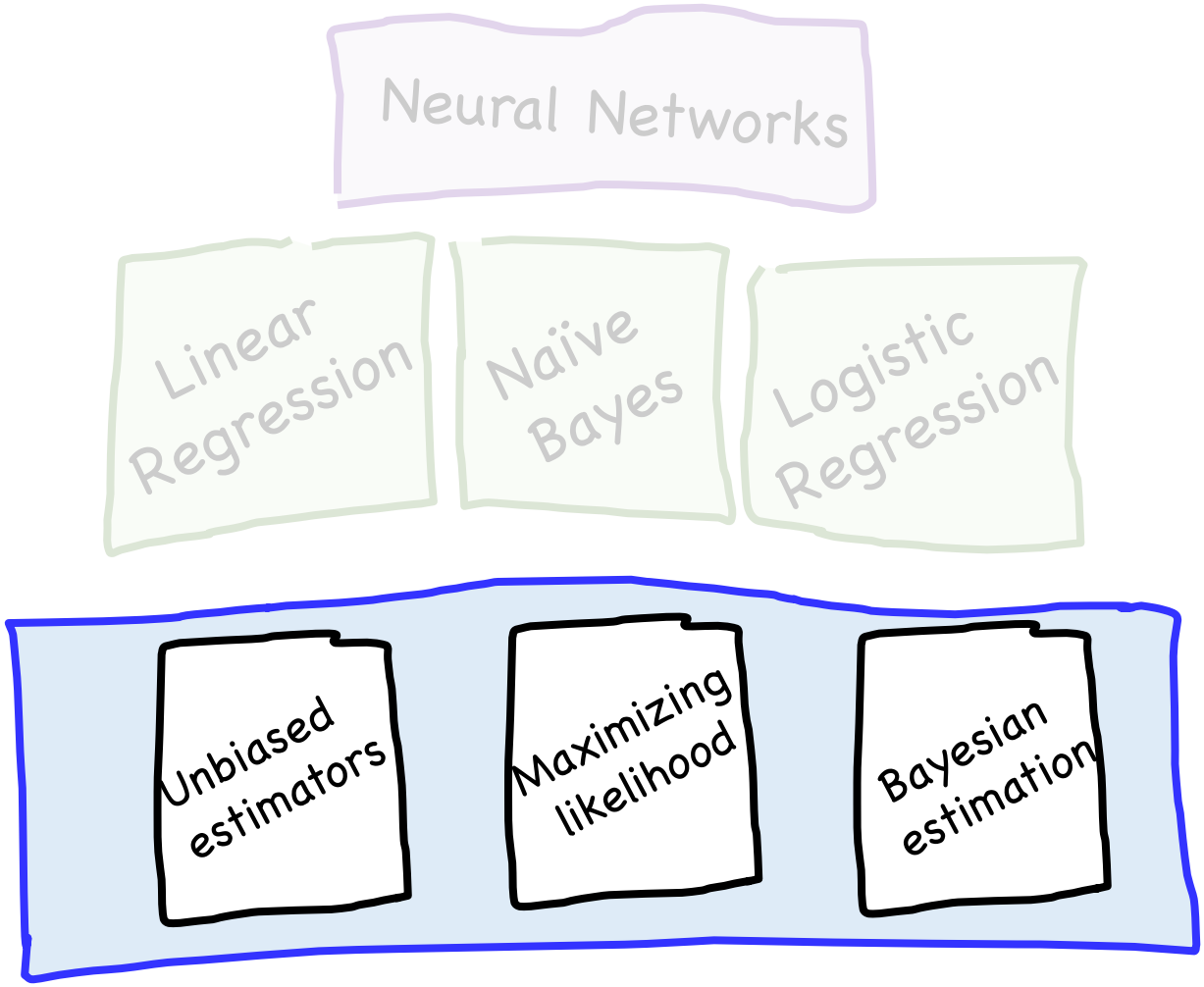
Parameter Estimation



Recall Sample Mean + Variance?

- Consider *n* I.I.D. random variables X_1, X_2, ... X_n
  - X_i have distribution *F* with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$
  - We call sequence of X_i a **sample** from distribution *F*
  - Recall sample mean: $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ where $E[\bar{X}] = \mu$

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \quad \text{as} \quad n \to \infty$$

  - Recall sample variance:

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1} = \text{undefined}$$

Estimate parameters for Bernoulli and Normal

Limited tool: how could we use that for fitting a mixture of gaussians?

Great idea in Machine Learning

Likelihood of Data

- Consider *n* I.I.D. random variables X_1, X_2, ..., X_n
  - X_i is a sample from density function $f(X_i \mid \theta)$
    - Note: now explicitly specify parameter θ of distribution

Likelihood question:
How likely is the data given the samples?

$$\text{Likelihood}(\theta) = f(\text{Samples}|\theta)$$

Demo


L-R
30
40
50
60

Likelihood of Data

- Consider *n* I.I.D. random variables X_1, X_2, ..., X_n
  - X_i is a sample from density function $f(X_i \mid \theta)$
    - Note: now explicitly specify parameter θ of distribution
  - We want to determine how "likely" the observed data (x_1, x_2, ..., x_n) is based on density $f(X_i \mid \theta)$
  - Define the **Likelihood function**, $L(\theta)$:

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

    - This is just a product since X_i are I.I.D.
  - Intuitively: what is probability of observed data using density function $f(X_i \mid \theta)$, for some choice of θ

Maximum Likelihood Estimator

- The **Maximum Likelihood Estimator** (MLE) of θ, is the value of θ that maximizes $L(\theta)$
  - More formally: $\theta_{MLE} = \arg\max_{\theta} L(\theta)$

Likelihood (of data given parameters):

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

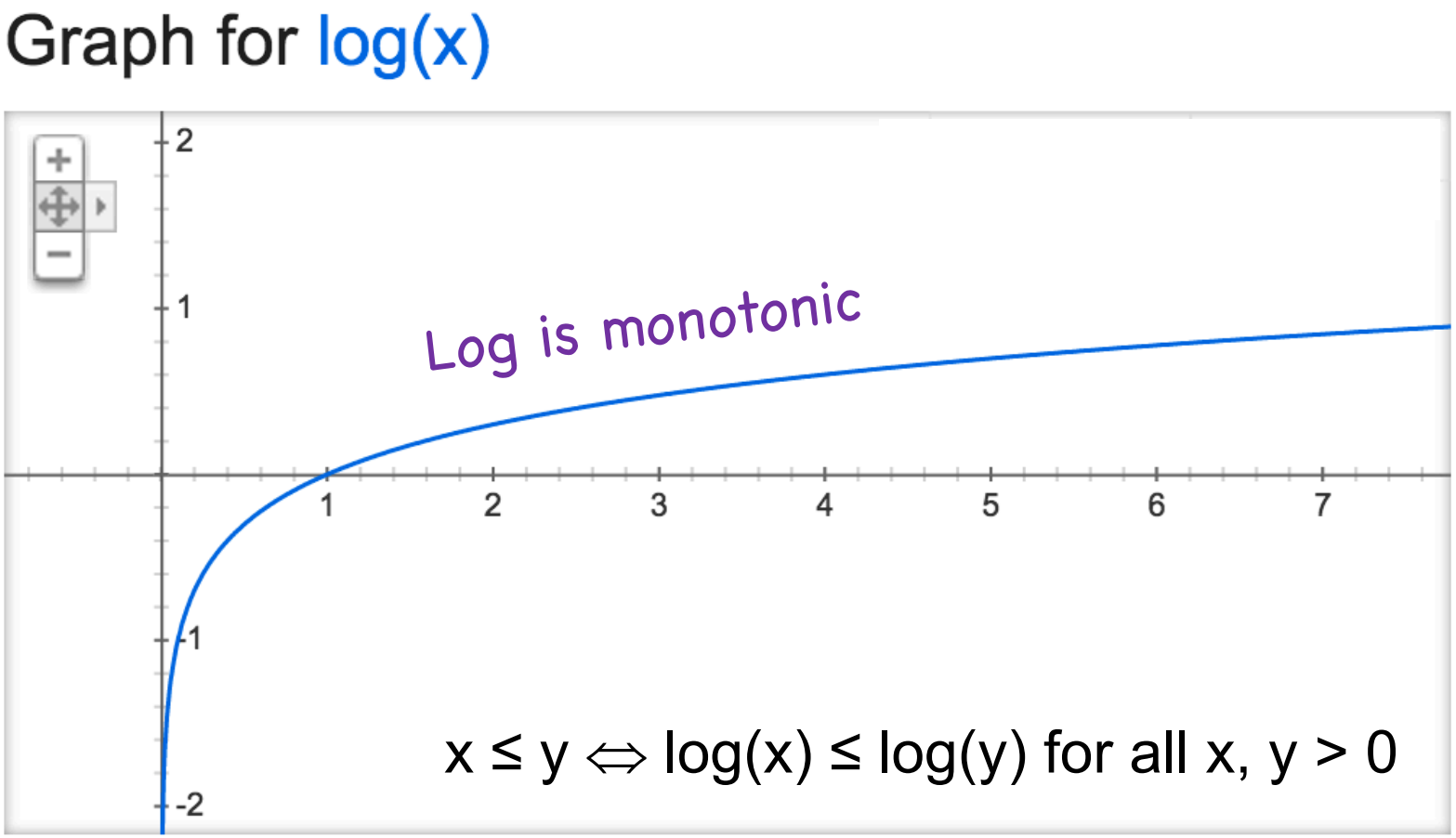
Argmax

$$f(x) = -x^2 + 5$$

$$\max_x \; -x^2 + 5 = 5$$

$$\operatorname*{argmax}_x \; -x^2 + 5 = 0$$

Argmax of Log



Claim: $\underset{x}{\operatorname{argmax}}\ f(x) = \underset{x}{\operatorname{argmax}}\ \log f(x)$

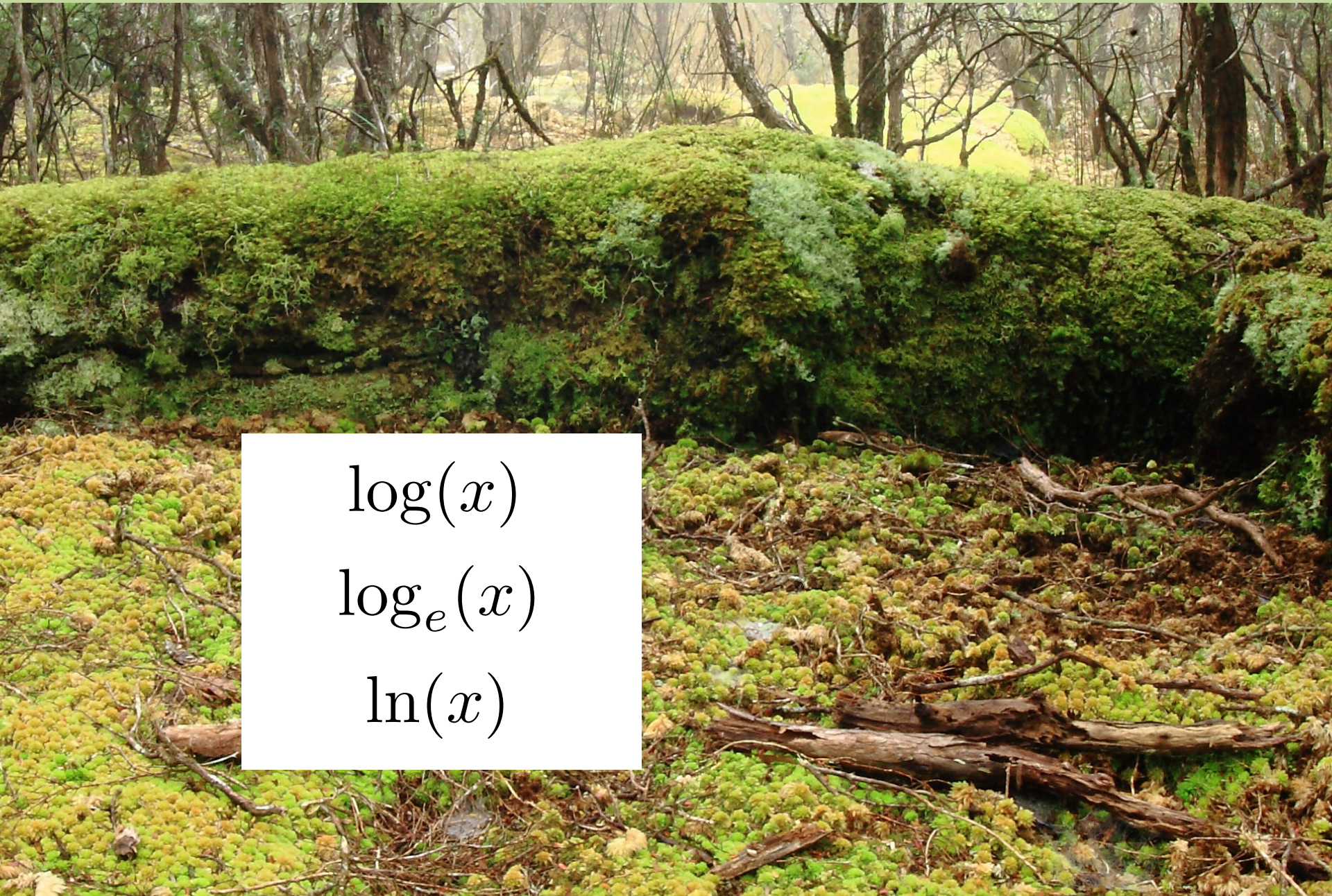
Argmax of Log

$$\underset{x}{\operatorname{argmax}}\ f(x) = \underset{x}{\operatorname{argmax}}\ \log f(x)$$

Log I Love You

$$\log(ab) = \log(a) + \log(b)$$

Natural Log



$$\log(x)$$

$$\log_e(x)$$

$$\ln(x)$$

Maximum Likelihood Estimator

- The **Maximum Likelihood Estimator** (MLE) of θ, is the value of θ that maximizes $L(\theta)$
  - More formally: $\theta_{MLE} = \arg\max_{\theta} L(\theta)$
  - More convenient to use **log-likelihood function**, $LL(\theta)$:

$$LL(\theta) = \log L(\theta) = \log \prod_{i=1}^{n} f(X_i \mid \theta) = \sum_{i=1}^{n} \log f(X_i \mid \theta)$$

  - Note that *log* function is "monotone" for positive values
    - Formally: $x \le y \Leftrightarrow \log(x) \le \log(y)$ for all $x, y > 0$
  - So, θ that maximizes $LL(\theta)$ also maximizes $L(\theta)$
    - Formally: $\arg\max_{\theta} LL(\theta) = \arg\max_{\theta} L(\theta)$
    - Similarly, for any positive constant c (not dependent on θ):

$$\arg\max_{\theta}(c \cdot LL(\theta)) = \arg\max_{\theta} LL(\theta) = \arg\max_{\theta} L(\theta)$$

Story so far: We can chose parameters by finding the argmax of the log likelihood of our data

Maximum Likelihood

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta)$$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ LL(\theta)$$

But how do we compute argmax?

Option #1: Straight optimization

Computing the MLE

- General approach for finding MLE of θ
  - Determine formula for $LL(\theta)$
  - Differentiate $LL(\theta)$ w.r.t. (each) θ : $\frac{\partial LL(\theta)}{\partial \theta}$
  - To maximize, set $\frac{\partial LL(\theta)}{\partial \theta} = 0$
  - Solve resulting (simultaneous) equations to get θ_{MLE}
    - Make sure that derived $\hat{\theta}_{MLE}$ is actually a maximum (and not a minimum or saddle point). E.g., check $LL(\theta_{MLE} \pm \varepsilon) < LL(\theta_{MLE})$
      - This step often ignored in expository derivations
      - So, we'll ignore it here too (and won't require it in this class)

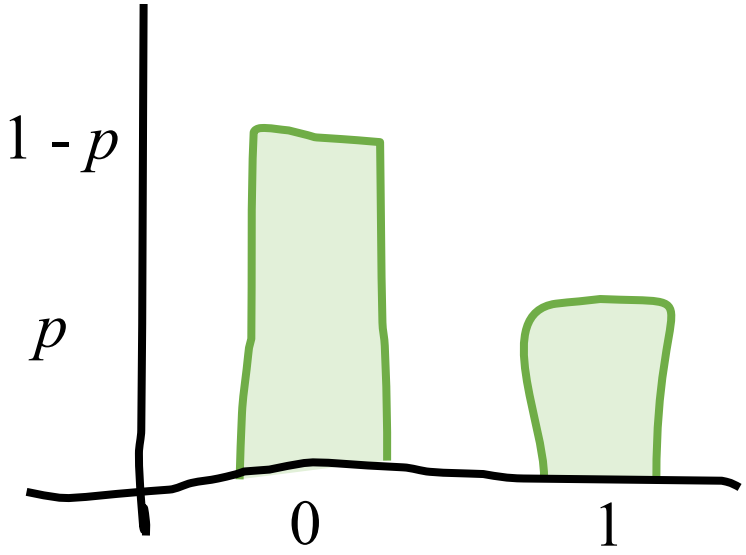
Maximizing Likelihood with Bernoulli

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Ber}(p)$
  - Probability mass function, $f(X_i \mid p)$:

Maximizing Likelihood with Bernoulli

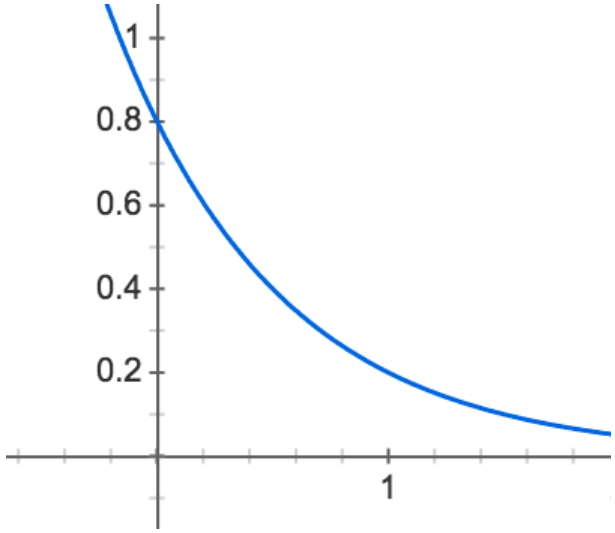
- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Ber}(p)$
  - Probability mass function, $f(X_i \mid p)$:

PMF of Bernoulli



$$f(X_i \mid p) = p^{x_i}(1-p)^{1-x_i}$$

PMF of Bernoulli (p = 0.2)



$$f(x) = 0.2^x(1-0.2)^{1-x}$$

Bernoulli PMF

$$X \sim \text{Ber}(p)$$

$$f(X = x|p) = p^x(1-p)^{1-x}$$

Maximizing Likelihood with Bernoulli

- Consider I.I.D. random variables X_1, X_2, ..., X_n
  - $X_i \sim \text{Ber}(p)$
  - Probability mass function, $f(X_i \mid p)$, can be written as:

$$f(X_i \mid p) = p^{x_i}(1-p)^{1-x_i} \quad \text{where } x_i = 0 \text{ or } 1$$

  - Likelihood: $L(\theta) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}$
  - Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} \log(p^{X_i}(1-p)^{1-X_i}) = \sum_{i=1}^{n} \left[X_i(\log p) + (1-X_i)\log(1-p)\right]$$
$$= Y(\log p) + (n-Y)\log(1-p) \quad \text{where } Y = \sum_{i=1}^{n} X_i$$

  - Differentiate w.r.t. p, and set to 0:

$$\frac{\partial LL(p)}{\partial p} = Y\frac{1}{p} + (n-Y)\frac{-1}{1-p} = 0 \quad \Rightarrow \quad p_{MLE} = \frac{Y}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Isn’t that the same as unbiased estimator?

Yes. For Bernoulli.

Maximum Likelihood Algorithm

1. Decide on a model for the distribution of your samples. Define the PMF / PDF for your sample.

2. Write out the log likelihood function.

3. State that the optimal parameters are the argmax of the log likelihood function.

4. Use an optimization algorithm to calculate argmax

Maximizing Likelihood with Poisson

- Consider I.I.D. random variables $X_1, X_2, \ldots, X_n$
  - $X_i \sim \text{Poi}(\lambda)$
  - PMF: $f(X_i \mid \lambda) = \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!}$ Likelihood: $L(\theta) = \prod_{i=1}^{n} \dfrac{e^{-\lambda}\lambda^{X_i}}{X_i!}$
  - Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n} \left[-\lambda \log(e) + X_i \log(\lambda) - \log(X_i!)\right]$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$$

  - Differentiate w.r.t. λ, and set to 0:

$$\frac{\partial LL(\lambda)}{\partial \lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} X_i = 0 \quad \Rightarrow \quad \lambda_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Its so general!

Maximizing Likelihood with Normal

- Consider I.I.D. random variables X_1, X_2, ..., X_n
  - $X_i \sim N(\mu, \sigma^2)$
  - PDF: $f(X_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}$
  - Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}\right) = \sum_{i=1}^{n} \left[-\log(\sqrt{2\pi}\sigma) - (X_i-\mu)^2/(2\sigma^2)\right]$$

  - First, differentiate w.r.t. μ, and set to 0:

$$\frac{\partial LL(\mu,\sigma^2)}{\partial \mu} = \sum_{i=1}^{n} 2(X_i-\mu)/(2\sigma^2) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu) = 0$$

  - Then, differentiate w.r.t. σ, and set to 0:

$$\frac{\partial LL(\mu,\sigma^2)}{\partial \sigma} = \sum_{i=1}^{n} -\frac{1}{\sigma} + 2(X_i-\mu)^2/(2\sigma^3) = -\frac{n}{\sigma} + \sum_{i=1}^{n}(X_i-\mu)^2/(\sigma^3) = 0$$

Being Normal, Simultaneously

- Now have two equations, two unknowns:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)=0 \qquad -\frac{n}{\sigma}+\sum_{i=1}^{n}(X_i-\mu)^2/(\sigma^3)=0$$

- First, solve for μ_{MLE}:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)=0 \;\Rightarrow\; \sum_{i=1}^{n}X_i=n\mu \;\Rightarrow\; \mu_{MLE}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

- Then, solve for σ^2_{MLE}:

$$-\frac{n}{\sigma}+\sum_{i=1}^{n}(X_i-\mu)^2/(\sigma^3)=0 \;\Rightarrow\; n\sigma^2=\sum_{i=1}^{n}(X_i-\mu)^2$$

$$\sigma^2_{MLE}=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_{MLE})^2$$

- Note: μ_{MLE} unbiased, but σ^2_{MLE} biased

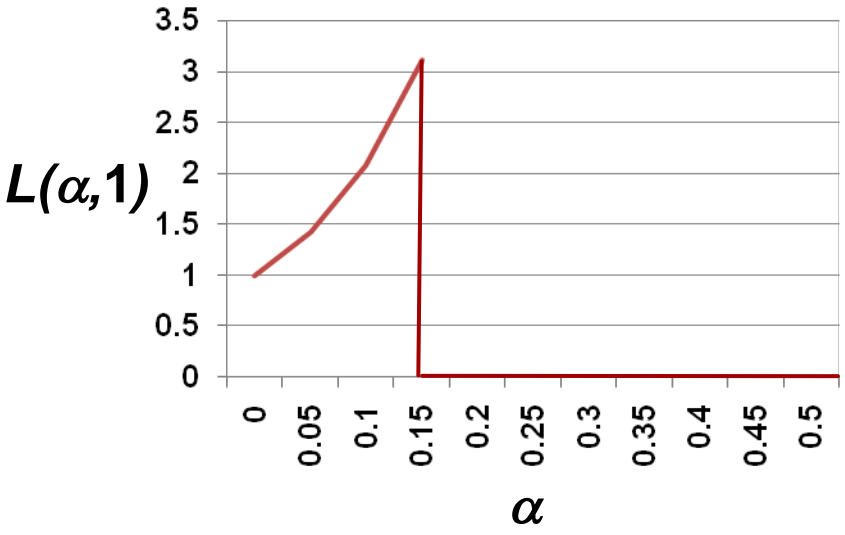
Maximizing Likelihood with Uniform

- Consider I.I.D. random variables $X_1, X_2, \ldots, X_n$
  - $X_i \sim \text{Uni}(\alpha, \beta)$
  - PDF: $f(X_i \mid \alpha, \beta) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \le x_i \le \beta \\ 0 & \text{otherwise} \end{cases}$
  - Likelihood: $L(\theta) = \begin{cases} \left(\frac{1}{\beta - \alpha}\right)^n & \alpha \le x_1, x_2, \ldots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$
    - Constraint $\alpha \le x_1, x_2, \ldots, x_n \le \beta$ makes differentiation tricky
    - Intuition: want interval size $(\beta - \alpha)$ to be as small as possible to maximize likelihood function for each data point
    - But need to make sure all observed data contained in interval
      - If all observed data not in interval, then $L(\theta) = 0$
  - Solution: $\alpha_{MLE} = \min(x_1, \ldots, x_n) \quad \beta_{MLE} = \max(x_1, \ldots, x_n)$

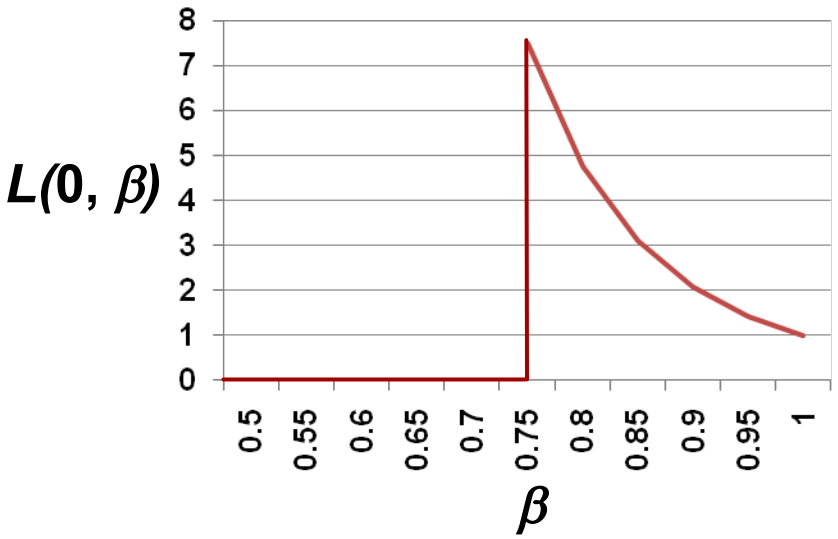
Understanding MLE with Uniform

- Consider I.I.D. random variables $X_1, X_2, \ldots, X_n$
  - $X_i \sim \text{Uni}(0, 1)$
  - Observe data:
    - 0.15, 0.20, 0.30, 0.40, 0.65, 0.70, 0.75

Likelihood: L(α,1)



Likelihood: L(0, β)



Small Samples = Problems

- How do small samples affect MLE?
  - In many cases, $\mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$ = sample mean
    - Unbiased. Not too shabby…
  - As seen with Normal, $\sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_{MLE})^2$
    - Biased. Underestimates for small *n* (e.g., 0 for n = 1)
  - As seen with Uniform, $\alpha_{MLE} \geq \alpha$ and $\beta_{MLE} \leq \beta$
    - Biased. Problematic for small *n* (e.g., $\alpha = \beta$ when n = 1)
  - Small sample phenomena intuitively make sense:
    - Maximum likelihood $\Rightarrow$ best explain data we've seen
    - Does not attempt to generalize to unseen data

Properties of MLE

- Maximum Likelihood Estimators are generally:
  - Consistent: $\lim_{n\to\infty} P(|\hat{\theta} - \theta| < \varepsilon) = 1$ for $\varepsilon > 0$
  - Potentially biased (though asymptotically less so)
  - Asymptotically optimal
    - Has smallest variance of "good" estimators for large samples
  - Often used in practice where sample size is large relative to parameter space
    - But be careful, there are some very large parameter spaces